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Formality and Deformations of Universal Enveloping Algebras

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Abstract We describe enveloping algebras of finite-dimensional Lie algebras which are formal in the sense that their Hochschild complex as a differential graded Lie algebra is quasi-isomorphic to its Hochschild cohomology. For Abelian Lie algebras this is true thanks to the Kontsevich formality theorem. We are using his formality map twisted by the group-like element generated by the linear Poisson structure to simplify the problem, and then study examples. For instance, the universal enveloping algebras of the Lie algebras $\mathfrak{gl}(n, \mathbb{K}) \oplus \mathbb{K}^n$ are formal. We also recover our rigidity results for enveloping algebras from this new angle and present some explicit deformations of linear Poisson structure in low dimensions.

1 Introduction

Since Maxim Kontsevich's seminal paper [29] on deformation quantization on any Poisson manifold, his concept of formality of associative algebras turned out to be extremely useful for the deformation theory of that algebra. An associative algebra is called formal if its Hochschild complex equipped with the Gerstenhaber graded Lie structure is quasi-isomorphic in the L_∞ sense to its Hochschild cohomology. If this is the case first order deformations having induced Gerstenhaber bracket equal to zero always integrate to formal deformations.

Kontsevich's basic example is the symmetric algebra of a vector space which is formal. An interesting playground for formality checks seems to be the class of universal enveloping algebras of Lie algebras which are very close to symmetric algebras. For these algebras it is also interesting to study their rigidity as an associative algebra, i.e. the question whether each formal associative deformation is equivalent to the zero deformation, see [2] for some

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results in that direction. We have called a Lie algebra strongly rigid if its enveloping algebra is rigid.

In this report we should like start an investigation of formality and rigidity of universal enveloping algebras of Lie algebras. We shall give a sufficient criterion for formality: if the Chevalley–Eilenberg complex of a Lie algebra with values in the symmetric algebra of the Lie algebra is formal, then the universal enveloping algebra itself is formal, see Theorem 6.2. The proof uses a twisting of the Kontsevich formality map which converges for linear Poisson structures. The above-mentioned Theorem can also be used to reprove the Nonrigidity Theorem of [2] by constructing nontrivial deformations of universal enveloping algebras by means of nontrivial deformations of the linear Poisson structure. Finally we present some explicit deformations of linear Poisson structures in low dimensions.

The paper is organized as follows: The Sect. 2 is dedicated to preliminaries on enveloping algebras and deformation theory. Then we recall in Sect. 3 the notion of strongly rigid Lie algebra which was introduced in [2] and show that they are contained in the class of rigid Lie algebras and their second scalar cohomology group must be 0. In Sect. 4, we discuss the formality maps for associative algebras. We recall the bialgebras structures and the Kontsevich formality from where the existence of formal deformations of formal associative algebras can be deduced. Also we twist the formality maps by grouplike elements in the associative algebras case (Sect. 4.3). The formality in the particular case of symmetric algebra is summarized in Sect. 5. Section 6 concerns the deformation and formality universal enveloping algebras through the linear Poisson structure with respect to the finite-dimensional Lie algebras. We show that the universal enveloping algebras of the Lie algebras of the affine groups of $\mathbb{K}^n$ are formal, a fact which do not find in the literature. Also we recover the nonrigidity theorem of universal enveloping algebra proved in [2], which states that the nontrivial deformation of the linear Poisson structure associated to the Lie algebra induces a nontrivial deformation of the universal enveloping algebra. In the Sect. 6, we construct examples of quadratic deformations of linear Poisson structure associated to n-dimensional Lie algebras for $n \leq 7$. Then we obtain a classification of n-dimensional strongly rigid solvable Lie algebras for $n \leq 6$. Therefore we have the following classification of n-dimensional strongly rigid Lie algebras ($n \leq 6$):

$$\{0\}; \quad \mathbb{C}; \quad \mathfrak{aff}(1,\mathbb{C}); \quad \mathfrak{sl}(2,\mathbb{C}); \quad \mathfrak{gl}(2,\mathbb{C}); \quad \mathfrak{aff}(1,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C});$$
$$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}); \quad \mathfrak{aff}(2,\mathbb{C}).$$

2 Preliminaries

2.1 Universal Enveloping Algebras

Let $\mathbb{K}$ be a commutative ring and $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Recall that a (left) $\mathfrak{g}$-representation of $\mathfrak{g}$ is a $\mathbb{K}$-module $\mathcal{M}$ and a $\mathbb{K}$-homomorphism

$$\begin{aligned} \mathfrak{g} \otimes \mathcal{M} &\mapsto \mathcal{M}, \\ x \otimes a &\mapsto xa \end{aligned} \tag{2.1}$$

such that $x(ya) - y(xa) = [x, y]a$. To each Lie algebra $\mathfrak{g}$, an associative $\mathbb{K}$-algebra $\mathcal{U}\mathfrak{g}$ is associated such that every (left) $\mathfrak{g}$-representation may be viewed as (left) $\mathcal{U}\mathfrak{g}$-representation and vice-versa. The algebra $\mathcal{U}\mathfrak{g}$ is constructed as follows.

Let $T\mathfrak{g}$ be the tensor algebra of the $\mathbb{K}$-module $\mathfrak{g}$, $T\mathfrak{g} = T^0 \oplus T^1 \oplus \cdots \oplus T^n \oplus \cdots$ where $T^n = \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (n times). In particular $T^0 = \mathbb{K}1$ and $T^1 = \mathfrak{g}$. The multiplication in $T\mathfrak{g}$ is the tensor product. Every $\mathbb{K}$-linear map $\mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}$ has a unique extension to a $T\mathfrak{g}$-module map $T\mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}$. If $\mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}$ is a $\mathfrak{g}$-module then the vector space $\mathfrak{g}$ inside $T\mathfrak{g}$ is in general not a Lie subalgebra being represented on $\mathcal{M}$. This is remedied if and only if the elements of $T\mathfrak{g}$ of the form $x \otimes y - y \otimes x - [x, y]$ where $x, y \in \mathfrak{g}$ are sent to 0. Consequently, one is led to introduce the two-sided ideal I generated by the elements $x \otimes y - y \otimes x - [x, y]$ where $x, y \in \mathfrak{g}$. The enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$ is thus defined as $T\mathfrak{g}/I$. It follows that $\mathfrak{g}$-representations and $\mathcal{U}\mathfrak{g}$-modules may be identified. Recall that every $\mathcal{U}\mathfrak{g}$-bimodule $\mathcal{M}$ is a $\mathfrak{g}$-module by $(x, m) \to xm - mx$, denoted by $\mathcal{M}_a$.

Assume that $\mathfrak{g}$ is a free Lie algebra. Let $\{x_i\}$ be a fixed basis of $\mathfrak{g}$ and y_i be the image of x_i by the map $\mathfrak{g} \to T\mathfrak{g} \to \mathcal{U}\mathfrak{g}$. We set $y_I = y_{i_1} \cdots y_{i_p}$ with I a finite sequence of indices $i_1, \ldots, i_p$ and $y_I = 1$ if $I = \emptyset$. The Poincaré–Birkhoff–Witt Theorem insures that the enveloping algebra $\mathcal{U}\mathfrak{g}$ is generated by the elements y_I corresponding to the increasing sequences I.

We denote by $\mathcal{S}V$ the symmetric algebra over a $\mathbb{K}$-module V. If $\mathbb{Q} \in \mathbb{K}$, then there exists a canonical bijection between $\mathcal{S}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}$ which is a $\mathfrak{g}$-module isomorphism between $\mathcal{S}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}_a$ [12, pp. 78–79].

We shall need to compute Hochschild and Chevalley–Eilenberg cohomology of $\mathcal{U}\mathfrak{g}$ and $\mathfrak{g}$, respectively, wherefore we shall cite the following two Theorems:

The classical Theorem due to H. Cartan et S. Eilenberg, [9, pp. 277] gives a link between the Hochschild cohomology of an enveloping algebra with values in an $\mathcal{U}\mathfrak{g}$-bimodule $\mathcal{M}$ (in particular $\mathcal{M} = \mathcal{U}\mathfrak{g}$) and the Chevalley–Eilenberg cohomology of the Lie algebra with values in the same module.

Theorem 2.1 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$. Then*

$$\mathbf{H}^n_H(\mathcal{U}\mathfrak{g}, \mathcal{M}) \simeq \mathbf{H}^n_{CE}(\mathfrak{g}, \mathcal{M}_a) \quad \forall n \in \mathbb{N}.$$

In particular, if $\mathbb{Q} \subset \mathbb{K}$

$$\mathbf{H}^n_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) \simeq \mathbf{H}^n_{CE}(\mathfrak{g}, \mathcal{U}\mathfrak{g}_a) \simeq \mathbf{H}^n_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}) \quad \forall n \in \mathbb{N}.$$

The *Hochschild–Serre Theorem* [28] gives the following factorization of the Chevalley–Eilenberg cohomology groups in the case of a decomposable solvable Lie algebra.

Theorem 2.2 *Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$ be a finite dimensional solvable Lie algebra over $\mathbb{K}$, where $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{g}$ and $\mathfrak{t}$ the supplementary subalgebra of $\mathfrak{n}$, reductive in $\mathfrak{g}$, such that the $\mathfrak{t}$-module induced on $\mathcal{U}_\mathbf{a}\mathfrak{g}$ is semisimple, then for all nonnegative integers p, we have*

$$\mathbf{H}^\mathbf{p}_\mathbf{CE}(\mathfrak{g}, \mathcal{U}_\mathbf{a}(\mathfrak{g})) \simeq \sum_{\mathbf{i+j=p}} \mathbf{H}^\mathbf{i}_\mathbf{CE}(\mathfrak{t}, \mathbb{K}) \otimes \mathbf{H}^\mathbf{j}_\mathbf{CE}(\mathfrak{n}, \mathcal{U}_\mathbf{a}\mathfrak{g})^\mathbf{t}$$

where $\mathbf{H}^j_{CE}(\mathfrak{n}, \mathcal{U}_\mathbf{a}\mathfrak{g})^\mathfrak{t}$ denotes the subspace of $\mathfrak{t}$-invariant elements.

2.2 Deformations and Cohomology

The formal deformation of rings and algebras was introduced by Gerstenhaber in 1964 [20]. He gave a tool to deform algebraic structures based on formal power series. The interest on deformations has grown with the development of quantum groups related to quantum mechanics [1]. Examples of quantum groups may be obtained as Hopf algebra deformations of the enveloping algebra of a Lie algebra. A. Fialowski et al. take another point of view for studying deformations: Instead of formal power series rings they consider more general commutative algebra [14–19, 31].

Unless otherwise stated, $\mathbb{K}$ denotes a field of characteristic 0. Let $\mathbb{K}[[t]]$ be the power series ring with coefficients in $\mathbb{K}$. For a $\mathbb{K}$-vector space E we denote by $E[[t]]$ the $\mathbb{K}[[t]]$-module of the power series with coefficients in E. Let $(\mathcal{A}, \mu_0)$ be an associative (resp. Lie) $\mathbb{K}$-algebra, then $(\mathcal{A}[[t]], \mu_0)$ is an associative (resp. Lie) $\mathbb{K}[[t]]$-algebra.

A *formal deformation* of an associative (resp. Lie algebra) $\mathcal{A}$ is an associative (resp. Lie) $\mathbb{K}[[t]]$-algebra $(\mathcal{A}[[t]], \mu)$ such that

$$\mu = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots + t^n\mu_n + \cdots,$$

where $\mu_n \in \mathbf{Hom}_{\mathbb{K}}(\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}, \mathcal{A})$ (resp. $\mu_n \in \mathbf{Hom}_{\mathbb{K}}(\mathcal{A} \wedge_{\mathbb{K}} \mathcal{A}, \mathcal{A})$). Moreover, two deformations $(\mathcal{A}[[t]], \mu)$ and $(\mathcal{A}[[t]], \mu')$ are said to be *equivalent* if there exists a formal isomorphism

$$\varphi = \varphi_0 + \varphi_1 t + \cdots + \varphi_n t^n + \cdots,$$

with $\varphi_0 = Id_{\mathcal{A}}$ (Identity map on $\mathcal{A}$) and $\varphi_n \in \mathbf{Hom}_{\mathbb{K}}(\mathcal{A}, \mathcal{A})$ such that

$$\mu'(a, b) = \varphi_t^{-1}(\mu(\varphi(a), \varphi(b))) \quad \forall a, b \in \mathcal{A}.$$

A deformation of $\mathcal{A}$ is called *trivial* if it is equivalent to $(\mathcal{A}[[t]], \mu_0)$. An associative (resp. Lie) algebra $\mathcal{A}$ is said to be *rigid* if every deformation of $\mathcal{A}$ is trivial.

Recall the relation of formal deformation theory to Hochschild cohomology in the case of an associative algebra and Chevalley–Eilenberg cohomology in the case of Lie algebra. We denote by $\mathbf{H}^n_H(\mathcal{A}, \mathcal{M})$ the n-th Hochschild cohomology group of an associative algebra $\mathcal{A}$ with values in the bimodule $\mathcal{M}$ and by $\mathbf{H}^n_{CE}(\mathfrak{g}, \mathcal{M})$ the n-th Chevalley–Eilenberg cohomology group of a Lie algebra $\mathcal{A}$ with values in a $\mathfrak{g}$-module $\mathcal{M}$. The second Hochschild cohomology group of an associative algebra (resp. Chevalley–Eilenberg cohomology group of a Lie algebra) with values in the algebra may be interpreted as the group of infinitesimal deformations. The rigidity Theorem of Gerstenhaber [20] (resp. of Nijenhuis–Richardson [32]) insures that if the 2nd Hochschild cohomology group $\mathbf{H}^2_H(\mathcal{A}, \mathcal{A})$ (resp. Chevalley–Eilenberg $\mathbf{H}^2_{CE}(\mathfrak{g}, \mathfrak{g})$) of an associative algebra $\mathcal{A}$ (resp. a Lie algebra $\mathfrak{g}$) vanishes then the algebra (resp. Lie algebra) is rigid. Therefore the semisimple associative (resp. Lie) algebras are rigid because their second cohomology groups are trivial [22].

The third cohomology group corresponds to the obstructions to extend a deformation of order n to a deformation of order $n + 1$ [20, 21, 32].

The rigidity of n-dimensional complex rigid Lie algebras was studied by Carles, Diakité, Goze and Ancochea-Bermudez. Carles and Diakité established the classification for $n \leq 7$ [6–8], and Ancochea with Goze did the classification for solvable Lie algebras for $n = 8$ and some classes [24]. The classification of associative rigid algebras are known up to $n \leq 6$ (see [30]).

3 Strongly Rigid Lie Algebras and Properties

We recall here the notion of strongly rigid Lie algebra introduced in [2]:

Definition 3.1 A Lie algebra $\mathfrak{g}$ is called strongly rigid if its enveloping algebra $\mathcal{U}\mathfrak{g}$ is rigid as an associative algebra.

The semisimple Lie algebras give examples of strongly rigid Lie algebras. In fact, the Whitehead lemmas induce that the first and second cohomology groups of a Lie algebra $\mathfrak{g}$ with values in every finite dimensional $\mathbb{K}$-module vanish. Therefore these Lie algebras are rigid as Lie algebra. Using the filtration of $\mathcal{S}\mathfrak{g}$ and the Cartan–Eilenberg Theorem we obtain $H_H^2(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) = 0$. Therefore, the enveloping algebra of a semisimple Lie algebra is rigid.

In the following, we show some properties and examples of strongly rigid Lie algebras.

3.1 Rigidity of the Lie Algebra

Theorem 3.1 *If $\mathfrak{g}$ is a finite dimensional strongly rigid Lie algebra over $\mathbb{K}$, then $\mathfrak{g}$ is rigid as a Lie algebra.*

Proof We suppose that the enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$ is rigid, but not the Lie algebra $\mathfrak{g}$. Then there exists a nontrivial formal deformation $(\mathfrak{g}[[t]], \mu)$ of $\mathfrak{g}$ with $\mu = \sum_{n=0}^{\infty} \mu_n t^n$ and the cohomology class of μ_1 is nontrivial in $\mathbf{H}^2_{CE}(\mathfrak{g}, \mathfrak{g})$. Since $\mathfrak{g}$ is finite dimensional, then the $\mathbb{K}[[t]]$-module $\mathfrak{g}[[t]]$ is isomorphic to the free module $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$. Let $y_I := y_{i_1} \cdots y_{i_k}$ be the generators of the PBW basis of $\mathcal{U}\mathfrak{g}$, let $y'_I := y'_{i_1} \bullet \cdots \bullet y'_{i_k}$ be the generators of PBW basis of $\mathcal{U}(\mathfrak{g}[[t]])$ over $\mathbb{K}[[t]]$ and that $\bullet$ is the multiplication in $\mathcal{U}(\mathfrak{g}[[t]])$. The map $\Phi : \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \to \mathcal{U}(\mathfrak{g}[[t]])$ defined by $\Phi(y_I) := y'_I$ is a $\mathbb{K}[[t]]$-module isomorphism. Let $\bar{\mu} : \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \times \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \to \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ the multiplication on the module $\mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ induced by $\bullet$ and Φ, i.e. $\bar{\mu}(a, b) := \Phi^{-1}(\Phi(a) \bullet \Phi(b))$. The restriction of $\bar{\mu}$ to elements of $\mathcal{U}\mathfrak{g} \times \mathcal{U}\mathfrak{g}$ defined a $\mathbb{K}$-bilinear map $\mathcal{U}\mathfrak{g} \times \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \subset (\mathcal{U}\mathfrak{g})[[t]]$ which we denote also by $\bar{\mu}$, i.e. $\bar{\mu}(u, v) = \sum_{n=0}^{\infty} t^n \bar{\mu}_n(u, v)$ for all $u, v \in \mathcal{U}\mathfrak{g}$ where $\bar{\mu}_n \in \mathbf{Hom}_{\mathbb{K}}(\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$. The $\mathbb{K}$-bilinear map $\bar{\mu}$ defined naturally a $\mathbb{K}[[t]]$-bilinear associative multiplication over the $\mathbb{K}[[t]]$-module $\mathcal{U}\mathfrak{g}[[t]]$ (which contains $\mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ as a dense submodule with respect to t-adic topology):

$$\bar{\mu}\left(\sum_{s}^{\infty} t^s u_s, \sum_{s'=0}^{\infty} t^{s'} v_{s'}\right) := \sum_{r=0}^{\infty} t^r \sum_{\substack{s,s',s''\geq 0 \\ s+s'+s''=r}} \bar{\mu}_{s''}(u_s, v_{s'}).$$

In particular, the map $\bar{\mu}_0$ defined an associative multiplication over the vector space $\mathcal{U}\mathfrak{g}$, and $(\mathcal{U}\mathfrak{g}[[t]], \bar{\mu})$ is a formal associative deformation of $(\mathcal{U}\mathfrak{g}, \bar{\mu}_0)$. For a finite increasing sequence I, J we have $\bar{\mu}_0(y_I, y_J) = \Phi^{-1}(y'_I \bullet y'_J)|_{t=0}$. By ordering the product $y'_I \bullet y'_J$ we obtain that $\bar{\mu}_0$ is the multiplication of $\mathcal{U}\mathfrak{g}$ and $(\mathcal{U}\mathfrak{g}[[t]], \bar{\mu})$ is a formal deformation of $\mathcal{U}\mathfrak{g}$. It follows that $\bar{\mu}_1$ is a Hochschild 2-cocycle of $\mathcal{U}\mathfrak{g}$, and the restriction of $\bar{\mu}_1$ to $X, Y \in \mathfrak{g}$ satisfies

$$\mu_1(X, Y) = \bar{\mu}_1(X, Y) - \bar{\mu}_1(Y, X) \quad \forall X, Y \in \mathfrak{g} \tag{3.1}$$

because the Lie algebra $(\mathfrak{g}[[t]], \mu)$ is a Lie subalgebra of $\mathcal{U}(\mathfrak{g}[[t]])$ which may be considered as an associative subalgebra of $(\mathcal{U}\mathfrak{g}[[t]], \bar{\mu})$. The rigidity of $\mathcal{U}\mathfrak{g}$ implies that there exists a formal isomorphism $\varphi = \sum_{r=0}^{\infty} \varphi_r t^r$, where $\varphi_0 = Id_{\mathcal{U}\mathfrak{g}}$ and $\varphi_n \in \mathbf{Hom}_{\mathbb{K}}(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ such that

$$\varphi_t(\bar{\mu}(u, v)) = \bar{\mu}(\varphi(u), \varphi(v)) \quad \forall u, v \in \mathcal{U}\mathfrak{g},$$

which is equivalent to

$$\sum_{r=0}^{\infty} t^r \sum_{\substack{a,b\geq 0\\ a+b=r}} \varphi_a(\bar{\mu}_b(u,v)) = \sum_{r=0}^{\infty} t^r \sum_{\substack{a,b,c\geq 0\\ a+b+c=n}} \bar{\mu}_a(\varphi_b(u),\varphi_c(v)) \quad \forall u,v\in\mathcal{U}\mathfrak{g}. \tag{3.2}$$

If $r=1$, the relation becomes

$$\bar{\mu}_1(u,v) = (\delta_H\varphi_1)(u,v) \quad \forall u,v\in\mathcal{U}\mathfrak{g} \tag{3.3}$$

where δ_H is a Hochschild coboundary operator (see [27]) with respect to the multiplication $\bar{\mu}_0$ of the enveloping algebra.

Then the formulae (3.1) and (3.3) imply

$$\begin{aligned}\mu_1(X,Y) &= (\delta_H\varphi_1)(X,Y) - (\delta_H\varphi_1)(Y,X)\\ &= X\varphi_1(Y) - \varphi(XY) + \varphi(X)Y - Y\varphi_1(X) + \varphi(YX) - \varphi(Y)X\\ &= (\delta_{CE}\varphi_1)(X,Y) \quad \forall X,Y\in\mathfrak{g}\end{aligned} \tag{3.4}$$

where δ_{CE} is the Chevalley–Eilenberg coboundary operator, (see [9]).

Therefore the class of μ_1 in $\mathbf{H}^2_{CE}(\mathfrak{g},\mathfrak{g})$ is trivial. Contradiction. □

This result shows that the class of strongly rigid Lie algebras is contained in the class of rigid Lie algebras.

3.2 Second Scalar Cohomology Group

In this section we give a necessary condition on the scalar Chevalley–Eilenberg cohomology group for the strong rigidity of a Lie algebra.

Let $\omega\in\mathbf{Z}^2_{CE}(\mathfrak{g},\mathbb{K})$ be a scalar 2-cocycle of the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_\omega=\mathfrak{g}\oplus\mathbb{K}c$ be a central extension of $\mathfrak{g}$ with ω such that the new bracket $[\ ,\ ]'$ is defined as usually by

$$[X+ac, Y+bc]' := [X,Y]+\omega(X,Y)c \quad \forall X,Y\in\mathfrak{g};\ a,b\in\mathbb{K}. \tag{3.5}$$

Theorem 3.2 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$ such that the second scalar cohomology group $\mathbf{H}^2_{CE}(\mathfrak{g},\mathbb{K})$ is different from 0, then $\mathfrak{g}$ is not strongly rigid.*

Proof Let $\omega\in\mathbf{Z}^2_{CE}(\mathfrak{g},\mathbb{K})$ be a 2-cocycle with a nonzero class and let $\mathfrak{g}_{t\omega}[[t]]$ be the one-dimensional central extension of the Lie algebra $\mathfrak{g}[[t]]=\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{K}[[t]]$ over $K=\mathbb{K}[[t]]$ (see (3.5)). The multiplication of the enveloping algebra $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$ of $\mathfrak{g}_{t\omega}[[t]]$ is denoted by $\bullet$. Let consider the two-sided ideal $\mathcal{I}:=(1-c')\bullet\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])=\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])\bullet(1-c')$ (where c' denote the image of c in $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$) and the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}:=\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])/\mathcal{I}$. Let $e_1,\ldots,e_n$ be the $\mathbb{K}$-basis of $\mathfrak{g}$. Then $c,e_1,\ldots,e_n$ is a $\mathbb{K}[[t]]$-basis of $\mathfrak{g}_{t\omega}[[t]]$. Let $y_1,\ldots,y_n$ be the images of the basis vectors in $\mathcal{U}\mathfrak{g}$ and $c',y'_1,\ldots,y'_n$ be the images of the basis vectors in $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$. Let $y_I:=y_{i_1}\cdots y_{i_k}$ in $\mathcal{U}\mathfrak{g}$ over $\mathbb{K}$ be the generators of the PBW basis. The elements $c'^{\bullet i_0}\bullet y'_I$ (where $i_0\in\mathbb{N}$ and $c'^{\bullet i_0}:=1$) form a basis of $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$ over $\mathbb{K}[[t]]$ (the Lie algebra is a free module over a commutative ring, see [9], p. 271). In the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}$, the element $c'^{\bullet i_0}$ is identified to 1. We denote the multiplication in $\mathcal{U}_{t\omega}\mathfrak{g}$ by $\cdot$ and by the canonical projection, the images of $y'_1,\ldots,y'_n$ by $y''_1,\ldots,y''_n$ the elements y'_I give $y''_I:=y''_{i_1}\cdot\ldots\cdot y''_{i_n}$. It follows that the elements y''_I form a basis of the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}$. As in the proof of the previous Theorem 3.1, the map $\Phi : \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \to \mathcal{U}_{t\omega}\mathfrak{g}$ given by $y_I \mapsto y''_I$ defines an isomorphism of free $\mathbb{K}[[t]]$-modules. In a similar way we show that the multiplication induced on $\mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ by the multiplication $\cdot$ of $\mathcal{U}_{t\omega}\mathfrak{g}$ and Φ define a sequence of $\mu = \sum_{r=0}^{\infty} \mu_r t^r$, where $\mu_r \in \mathbf{Hom}_{\mathbb{K}}(\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ with the following properties: 1. μ defines a formal associative deformation of $(\mathcal{U}\mathfrak{g}, \mu_0)$, 2. μ_0 is the usual multiplication of the enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$. Therefore, μ_1 is a Hochschild 2-cocycle of $\mathcal{U}\mathfrak{g}$, and for all $X, Y \in \mathfrak{g} \subset \mathcal{U}\mathfrak{g}$ we have the relation: $\omega(X, Y)1 = \mu_1(X, Y) - \mu_1(Y, X)$ because the Lie algebra $\mathfrak{g}_{t\omega}[[t]]$ is injected in the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}$, then in $\mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \subset \mathcal{U}\mathfrak{g}[[t]]$.

Suppose that $\mathcal{U}\mathfrak{g}$ is rigid, then the deformation μ is trivial. Therefore there exists a Hochschild 1-cocycle $\varphi_1 \in \mathbf{C}^1_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ such that $\mu_1 = \delta_H(\varphi_1)$. It follows $\forall X, Y \in \mathfrak{g}$:

$$\begin{aligned}\omega(X, Y)1 &= \mu_1(X, Y) - \mu_1(Y, X) = \delta_H(\varphi_1)(X, Y) - \delta_H(\varphi_1)(Y, X)\\ &= \delta_{CE}(\varphi_1)(X, Y).\end{aligned}$$

Then ω is a Chevalley–Eilenberg coboundary and its class is trivial in $\mathbf{H}^2_{CE}(\mathfrak{g}, \mathbb{K})$, contradiction. □

3.3 Examples

The previous Theorems allow us to show that some classes of solvable Lie algebras are not strongly rigid.

Corollary 3.1 *The following Lie algebras are not strongly rigid*:

1. *Every n-dimensional nilpotent Lie algebra $\mathfrak{g}$ with n greater or equal than* 2.
2. *Every Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ where the dimension of the torus $\mathfrak{t}$ is greater or equal than* 2.

Proof The first assertion is a consequence of a classical result of Dixmier concerning the nilpotent Lie algebras [11]: $\mathbf{H}^2_{CE}(\mathfrak{g}, \mathbb{K}) \neq 0$ if $\dim(\mathfrak{g} \geq 2)$.

For the second, we have $\mathbf{H}^2_{CE}(\mathfrak{t}, \mathbb{K}) \neq \{0\}$ for an Abelian Lie subalgebra implying that $\mathbf{H}^2_{CE}(\mathfrak{g}, \mathbb{K}) \neq \{0\}$ is nonzero by the Hochschild–Serre Theorem 2.2. □

In the following we show, by an elementary proof, that the 2-dimensional non Abelian Lie algebra is strongly rigid. We denote by $\mathfrak{r}_2$ the solvable Lie algebra generated by X, Y such that $[X, Y] = Y$.

Lemma 3.1

1. $\forall n, m \in \mathbb{N} : YX^n = (X-1)^n Y$, $[X, Y^m] = mY^m$ *and* $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}^*$ $(m-1)X^nY^m = [X, X^nY^m] - X^nY^m$.
2. *There exists a polynomial $P_{n+1}(X)$ in X of degree $n+1$ such that*:
   (a) $P_1(X) = X$ *and* $P_{n+1}(X) = X^{n+1} + \sum_{k=2}^{n+1}(-1)^k\binom{n+2}{k}P_{n+2-k}(X)$, *if* $n \geq 1$.
   (b) $(n+1)X^nY = [P_{n+1}(X), Y]$.

Proof The first assertion may easily be proved by induction.

Let us prove the property (2) by induction on n.

It is true for $n=0$, because $[P_1(X),Y]=[X,Y]=Y$. Assume that it is true until n. We have (a):

$$\begin{aligned}[X^{n+2},Y] &= X^{n+2}Y - YX^{n+2}\\ &= X^{n+2}Y-(X-1)^{n+2}Y \quad \text{following (1)}\\ &= X^{n+2}Y-\sum_{k=0}^{n+2}(-1)^k\binom{n+2}{k}X^{n+2-k}Y\\ &= (n+2)X^{n+1}Y-\sum_{k=2}^{n+2}(-1)^k\binom{n+2}{k}X^{n+2-k}Y.\end{aligned}$$

Applying the induction hypothesis on $n+2-k$ with $k\geq 2$, we obtain $X^{n+2-k}Y=[P_{n+3-k}(X),Y]$ (the degree of $P_{n+3-k}(X)=n+3-k\leq n+1$).

Then (b) becomes

$$(n+2)X^{n+1}Y=\left[X^{n+2}+\sum_{k=2}^{n+2}(-1)^k\binom{n+2}{k}P_{n+3-k}(X),Y\right]=[P_{n+2}(X),Y]. \qquad \square$$

Proposition 3.1 *Let $\mathfrak{r}_2$ be the 2-dimensional non Abelian Lie algebra. We have*

$$\mathbf{H}^2_H(\mathcal{U}\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)\simeq \mathbf{H}^2_{CE}(\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)=0.$$

Thus, the Lie algebra $\mathfrak{r}_2$ is strongly rigid.

Proof By the Cartan–Eilenberg Theorem we have

$$\mathbf{H}^2_H(\mathcal{U}\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)\simeq \mathbf{H}^2_{CE}(\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2).$$

We show that

$$\forall \Phi\in \mathbf{Z}^2_{CE}(\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)\ \exists f\in \mathbf{C}^1_{CE}(\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)\quad \text{s.t.}\quad \delta_{CE}(f)=\Phi. \qquad (*)$$

Let $\{X^nY^m : n,m\in\mathbb{N}\}$ be the Poincaré–Birkhoff–Witt basis of $\mathcal{U}\mathfrak{r}_2$. Let Φ be an element of $\mathbf{Z}^2_{CE}(\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)$. It is defined by $\Phi(X,Y)=:u=:\sum_{n,m\in\mathbb{N}}u_{n,m}X^nY^m$ where $u_{n,m}\in\mathbb{K}$ are nonzero for a finite number of n,m. Let f be an element of $\mathbf{C}^1_{CE}(\mathfrak{r}_2,\mathcal{U}\mathfrak{r}_2)$. It is defined by two elements $f(X)=:v=:\sum_{n,m\in\mathbb{N}}v_{n,m}X^nY^m$ and $f(Y)=w=\sum_{n,m\in\mathbb{N}}w_{n,m}X^nY^m$ where $v_{n,m},w_{n,m}\in\mathbb{K}$ are nonzero for a finite number of n,m.

Then

$$\forall u=\sum_{n,m\in\mathbb{N}}u_{n,m}X^nY^m\in\mathcal{U}\mathfrak{r}_2\ \exists v,w\in\mathcal{U}\mathfrak{r}_2\quad\text{such that}\quad u=[X,w]-w+[v,Y]. \qquad (**)$$

We consider two cases:

Case 1: $m\neq 1$.

We set $w_{n,m}=\frac{u_{n,m}}{m-1}$, then $v_{n,m}=0$ if $m\neq 1$ and v,w satisfy $(**)$ by lemma (3.1,(2)).

Case 2: $m=1$.

We set $v_{n,m}=\frac{1}{n+1}u_{n,1}P_{n+1}(X)$ then $w_{n,m}=0$ if $m=1$ and v,w satisfy $(**)$ by lemma (3.1,(1)).

We conclude that the relation ($**$) is satisfied. Therefore $\mathbf{H}^2_{CE}(\mathfrak{r}_2, \mathcal{U}\mathfrak{r}_2) = 0$, and the Lie algebra $\mathfrak{r}_2$ is strongly rigid. □

In general for the *affine Lie algebras* $\mathfrak{aff}(m, \mathbb{K}) := \mathfrak{gl}(m, \mathbb{K}) \oplus \mathbb{K}^m$ (semidirect sum) for $m \in \mathbb{N} \setminus \{0\}$ we have

Proposition 3.2 *We have*

$$\forall k \in \mathbb{N}: \quad \mathbf{H}^k_{CE}(\mathfrak{aff}(m, \mathbb{K}), \mathcal{S}\mathfrak{aff}(m, \mathbb{K})) \cong \mathbf{H}^k_{CE}(\mathfrak{gl}(m, \mathbb{K}), \mathbb{K})$$

whence in particular $\mathbf{H}^2_{CE}(\mathfrak{aff}(m, \mathbb{K}), \mathcal{S}\mathfrak{aff}(m, \mathbb{K}))$ *vanishes and the affine Lie algebras are strongly rigid.*

See [2] for the proof.

4 Formality Maps for Associative Algebras

In this section we shall summarize some results on formality, see also [4, 26, 29].

4.1 Graded Bialgebras

Let $\mathbb{K}$ be a field of characteristic 0. Recall that a $\mathbb{Z}$*-graded vector space* V is a direct sum $\oplus_{i\in\mathbb{Z}} V^i$ of subspaces V_i. An element x of V lying in one of the V_i is called homogeneous, and we shall denote by $i =: |x| \in \mathbb{Z}$ its degree. Given two graded vector spaces V and W, a linear map $\phi : V \to W$ is said to be homogeneous of degree j if and only if for all integers i we have $\phi(V^i) \subset W^{i+j}$. In graded situations we shall write $\mathrm{Hom}(V, W)^j$ for the vector space of all linear maps which are homogeneous of degree j, and $\mathrm{Hom}(V, W)$ for the direct sum of all the $\mathbf{Hom}(V, W)^j$. Clearly, $\mathbf{Hom}(V, W)$ is a graded vector space. Likewise, the tensor product $V \otimes W$ is graded by setting $(V \otimes W)^i = \oplus_{a\in\mathbb{Z}} V^a \otimes W^{i-a}$. Recall the Koszul rule of signs: let $\phi : V \to W$ and $\psi : V' \to W'$ two homogeneous linear maps. Then for all homogeneous elements $x \in V$ and $y \in V'$

$$(\phi \otimes \psi)(x \otimes y) := (-1)^{|\psi||x|}\phi(x) \otimes \psi(y)$$

which defines the graded tensor product of linear maps. Recall that a graded (associative) algebra $(\mathcal{A}, \mu)$ is a graded vector space $\mathcal{A}$ together with a linear map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ of degree 0. Very often we write aa' for $\mu(a \otimes a')$. For another graded algebra $(\mathcal{B}, \nu)$ their graded tensor product is defined for homogeneous elements $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ by

$$(a \otimes b)(a' \otimes b') := (-1)^{|b||a'|} aa' \otimes bb'.$$

For an integer j denote by $V[j]$ the shifted graded vector space defined by $V[j]^i := V^{i+j}$. The identity map $V \to V$ induces a map of degree $-j : V \to V[j]$. It is called *suspension map* s for the case $s : V \to V[-1]$ whence it is of degree one. The map $V \to V[j]$ is considered as the $-j^{\text{th}}$ power of s. Multilinear maps $\phi : V^{\otimes k} \to W$ can be shifted, i.e. $\phi[j] : V[j]^{\otimes k} \to W[j]$ by setting $\phi[j] := s^{-j} \circ \phi \circ (s^{\otimes k})^j$. The degree of $\phi[j]$ is given by $|\phi[j]| = j(k-1) + |\phi|$.

Recall the *graded symmetric bialgebra* $\mathcal{S}V$ of a $\mathbb{Z}$-graded vector space V: In the free algebra TV over V (which inherits its $\mathbb{Z}$-grading by the $\mathbb{Z}$-grading of V) divide out the two-sided ideal generated by all elements $xy - (-1)^{|x||y|}yx$ in TV with homogeneous $x, y \in V$. The resulting multiplication $\bullet$ is graded commutative, i.e. for two homogeneous elements $a, b \in \mathcal{S}V$ we have $a \bullet b = (-1)^{|a||b|} b \bullet a$. Note also that on TV there is the graded shuffle comultiplication $\Delta : TV \to TV \otimes TV$ defined on generators by $\Delta(x) = x \otimes 1 + 1 \otimes x$ which is graded cocommutative. Δ passes to the quotient, and equips $\mathcal{S}V$ with the structure of a bialgebra. The graded symmetric algebra $\mathcal{S}V$ is a graded free algebra in the sense that any linear map of degree 0 of V into another graded commutative algebra $\mathcal{A}$ can uniquely be extended to an algebra map $\mathcal{S}V \to \mathcal{A}$, and conversely any such algebra map is uniquely determined by its restriction to V. Likewise, any homogeneous linear map $d : V \to \mathcal{S}V$ can uniquely be extended to a graded derivation of $\mathcal{S}V$ and conversely, any such derivation is uniquely determined by its restriction to V. The comultiplication, however, is more important for formality. It turns out that $\mathcal{S}V$ is a graded commutative cofree coalgebra in the following sense: Consider the category $\mathcal{C}_{AN}$ of all such graded commutative coalgebras C that have exactly one grouplike element 1 and whose quotient coalgebra $C/\mathbb{K}1$ is conilpotent in the sense that for each element there is an integer n such that the n-fold iterated comultiplication vanishes on it. For any such coalgebra C denote by C^+ the kernel of the counit map ϵ, so clearly $C = \mathbb{K}1 \oplus C^+$. Then for each homogeneous linear map ϕ of degree zero of C^+ into V there is a unique map of graded coalgebras $\Phi : C \to \mathcal{S}V$ such that $\mathsf{pr}_V \circ \Phi = \phi$ where $\mathsf{pr}_V : \mathcal{S}V \to V$ is the canonical projection. Φ can be computed to be $\Phi = e^{*\phi}$ where $*$ is the convolution product in the space $\mathrm{Hom}(C, \mathcal{S}V)$. Conversely, any such coalgebra map Φ is uniquely determined by its component $\mathsf{pr}_V \circ \Phi$. Likewise, for any homogeneous linear map $d : \mathcal{S}V \to V$ there is a unique graded coderivation $D : \mathcal{S}V \to \mathcal{S}V$ such that $d = \mathsf{pr}_V \circ D$ which can be computed to be $D = d * \mathsf{id}$ using convolutions. Conversely, any graded coderivation D of $\mathcal{S}V$ is determined by its component $\mathsf{pr}_V \circ D$.

4.2 Kontsevich Formality

Let $(\mathcal{A}, \mu_0)$ be an associative algebra over the field $\mathbb{K}$. For each positive integer n let $\mathfrak{A}^n := \mathbf{C}^n_H(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathcal{A})$ and $\mathfrak{A}^0 := \mathcal{A}$ the space of Hochschild cochains of degree n. Set $\mathfrak{A} := \oplus_{k=0}^{\infty} \mathfrak{A}^k$ which is thus a $\mathbb{Z}$-graded vector space upon defining $\mathfrak{A}^k := \{0\}$ for strictly negative integers k. Recall that the *Gerstenhaber multiplication* $\circ_G : \mathbf{C}^k_H(\mathcal{A}, \mathcal{A}) \times \mathbf{C}^l_H(\mathcal{A}, \mathcal{A}) \to \mathbf{C}^{k+l-1}_H(\mathcal{A}, \mathcal{A})$ is defined by

$$\begin{aligned}(f \circ_G g)&(a_1, \ldots, a_{k+l-1}) \\ &= \sum_{i=1}^{k} (-1)^{(i-1)(l-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+l-1}), a_{i+l}, \ldots, a_{k+l-1}), \qquad (4.1)\end{aligned}$$

and the *Gerstenhaber bracket* is given by $[f, g]_G := f \circ_G g - (-1)^{(k-1)(l-1)} g \circ_G f$. It follows that the shifted space $(\mathfrak{A}[1], [\ ,\ ]_G)$ is a graded Lie algebra. Note that μ is associative if and only if $[\mu, \mu]_G = 0$. In that case the square of $b := [\mu,\ ]_G$ vanishes and defines up to a global sign the Hochschild coboundary operator δ_H.

For each nonnegative integer k denote by $\mathfrak{a}^k$ the kth Hochschild cohomology group

$$\mathfrak{a}^k := \mathbf{H}^k_H(\mathcal{A}, \mathcal{A}) := \frac{\mathrm{Ker}(b : \mathfrak{A}^k \to \mathfrak{A}^{k+1})}{\mathrm{Im}(b : \mathfrak{A}^{k-1} \to \mathfrak{A}^k)}$$

where again we set $\mathfrak{a}^k := \{0\}$ for all strictly negative integers k. The graded Jacobi identity implies that the Gerstenhaber bracket descends to a graded Lie-bracket $[\ ,\ ]_s$ on the shifted cohomology space $\mathfrak{a}[1]$.

We shall call a *generalized HKR-map* any graded linear injection ϕ of degree 0 of the Hochschild cohomology $\mathfrak{a}$ into the subspace of cocycles of $\mathfrak{A}$, i.e. $b \circ \phi = 0$. By elementary linear algebra this is always possible. However, in general ϕ will not be a morphism of graded Lie algebras $(\mathfrak{a}[1], [\ ,\ ]_s) \to (\mathfrak{A}[1], [\ ,\ ]_G)$: by construction one only has for two classes $f, g \in \mathfrak{a}$

$$\phi[f, g]_s = [\phi(f), \phi(g)]_G + b(\Phi_2(f, g))$$

so ϕ is a Lie algebra morphism up to a coboundary $b(\Phi_2(f, g))$. One may hope that this process can be continued for higher order terms. In a more algebraic manner: consider the shifted spaces $\mathfrak{a}[2]$ and $\mathfrak{A}[2]$. Consider the shifted maps $b[1] = b$ and $[\ ,\ ]_G[1]$ on $\mathfrak{A}[2]$ and $[\ ,\ ]_s[1]$ on $\mathfrak{a}[2]$. Thanks to the shift it follows that both $[\ ,\ ]_G[1]$ and $[\ ,\ ]_s[1]$ are graded *symmetric* maps, i.e. $[\ ,\ ]_G[1]$ is a degree 1 map from $\mathcal{S}^2(\mathfrak{A}[2]) \to \mathfrak{A}[2]$ and $[\ ,\ ]_s[1]$ is a degree 1 map from $\mathcal{S}^2(\mathfrak{a}[2]) \to \mathfrak{a}[2]$. Let $d : \mathcal{S}(\mathfrak{a}[2]) \to \mathcal{S}(\mathfrak{a}[2])$ be the unique coderivation of $\mathcal{S}(\mathfrak{a}[2])$ induced by $[\ ,\ ]_s[1]$, and let $b + D$ be the unique coderivation of $\mathcal{S}(\mathfrak{A}[2])$ induced by $b + [\ ,\ ]_G[1]$. Thanks to the graded Jacobi identity of both graded Lie brackets and the structure of b as a graded commutator it follows that $d^2 = 0$ and $(b + D)^2 = 0$.

Definition 4.1 The associative algebra $(\mathcal{A}, \mu_0)$ is called *formal* if and only if there is a morphism of differential graded coalgebras (of degree 0) $\Phi : \mathcal{S}(\mathfrak{a}[2]) \to \mathcal{S}(\mathfrak{A}[2])$, i.e.

$$(\Phi \otimes \Phi) \circ \Delta_{\mathcal{S}(\mathfrak{a}[2])} = \Delta_{\mathcal{S}(\mathfrak{A}[2])} \circ \Phi \quad \text{and} \quad (b + D) \circ \Phi = \Phi \circ d, \tag{4.2}$$

such that the restriction of Φ to $\mathfrak{a}[2]$ is a HKR map. Φ is called a formality map or an L_∞-morphism.

The above discussion has shown that Φ is determined by its components $\Phi_k : \mathcal{S}^k(\mathfrak{a}[2]) \to \mathfrak{A}[2]$, which remedy order-by-order the above mentioned failure of the HKR-map Φ_1 to be a morphism of graded Lie algebras (L_∞-algebra).

The formal deformation theory of a formal associative algebra $(\mathcal{A}, \mu_0)$ is very simple as Kontsevich has shown: let $\pi \in \mathbf{H}^2_H(A, A)[[h]] = \mathfrak{a}^2[[h]] = \mathfrak{a}[2]^0[[h]]$. Suppose that

$$[\pi, \pi]_s = 0. \tag{4.3}$$

Then it is always possible to construct a formal associative deformation $\mu = \mu_0 + \mu_*$ where $\mu_* := \sum_{r=1}^\infty h^r \mu_r$ such that the cohomology class $[\mu_1]$ of μ_1 is equal to π:

Consider $\mathcal{S}(\mathfrak{a}[2])[[h]]$ and $\mathcal{S}(\mathfrak{A}[2])[[h]]$ as topological bialgebras (with respect to the h-adic topology) with the canonical extension of all the structure maps. Note that the tensor product is no longer algebraic, but given by $(\mathcal{S}(\mathfrak{a}[2]) \otimes \mathcal{S}(\mathfrak{a}[2]))[[h]]$. Let $\bullet$ denote the shuffle-multiplication in a graded symmetric algebra. For a general graded vector space V it can be easily seen that the group-like elements of $\mathcal{S}(V)[[h]]$ are no longer exclusively given by 1, but by exponential functions of any primitive elements of degree zero, i.e. they take the form $e^{\bullet hv}$ with $v \in V^0[[h]]$. The image of the grouplike element $e^{\bullet h\pi}$ in $S(\mathfrak{a}[2])[[h]]$ under the formality map Φ, $\Phi(e^{\bullet h\pi})$ is a grouplike element in $\mathcal{S}(\mathfrak{A}[2])[[h]]$ and thus takes the form $e^{\bullet \mu_*}$ with $\mu_* \in h\mathfrak{A}^2[[h]]$. Since $[\pi, \pi]_s = 0$ it follows that $d(e^{\bullet h\pi}) = 0$, and therefore $(b +$

$D)(e^{\bullet h\mu_*}) = 0$. Projecting this last equation to $\mathfrak{A}[2]^0[[h]] = \mathfrak{A}^2[[h]]$, we get the *Maurer–Cartan equation*

$$0 = b\mu_* + \frac{1}{2}[\mu_*, \mu_*]_G = \frac{1}{2}[\mu_0 + \mu_*, \mu_0 + \mu_*]_G,$$

showing the associativity of $\mu = \mu + \mu_*$. Hence $\mu := \mu_0 + \mu_*$ is a formal associative deformation of the algebra (A, μ_0).

4.3 Twisted Formality Maps for Associative Algebras

In this section we show that formality maps can be twisted by grouplike elements, compare also [23].

Lemma 4.1 *Let $(C, \tilde{\mu}, 1, \Delta, \epsilon, S)$ and $(C', \tilde{\mu}', 1', \Delta', \epsilon', S')$ be graded topological Hopf algebras. Let $u \in C$ be a group-like element and $p \in C$ be a primitive element of degree $|p|$.*

1. *If $\Phi : C \to C'$ is a morphism of topological graded coalgebras then $\hat{\Phi} : C \to C' : x \mapsto \Phi(u)^{-1}\Phi(ux) =: \hat{\Phi}(x)$ is again a morphism of topological graded coalgebras.*
2. *If $D : C \to C$ is a graded coderivation, then $\hat{D} : C \to C$ defined by $\hat{D}(x) := u^{-1}D(ux)$ is also a graded coderivation of the same degree. In case $D^2 = 0$ then $\hat{D}^2 = 0$.*
3. *Moreover, for a graded coderivation D of degree $|D|$ the map $\tilde{D}(x) := p(D(x)) - (-1)^{|p|\,|D|}D(px)$ is a coderivation of C of degree $|D| + |p|$.*

Proof We shall denote by L_a left multiplication by a in C or C'. For any morphism ϕ of associative graded topological algebras we clearly have $\phi \circ L_a = L_{\phi(a)} \circ \phi$. Now, u clearly is of degree 0 and $\Phi(u)$ is also grouplike and invertible by the existence of an antipode.

1. We have $\hat{\Phi} = L_{\Phi(u)^{-1}} \circ \Phi \circ L_u$, and we get

$$\begin{aligned}\Delta' \circ \hat{\Phi} &= \Delta' \circ L_{\Phi(u)^{-1}} \circ \Phi \circ L_u = L_{(\Delta'(\Phi(u)))^{-1}} \circ (\Phi \otimes \Phi) \circ L_{\Delta(u)} \circ \Delta \\ &= (L_{\Phi(u)^{-1}} \otimes L_{\Phi(u)^{-1}}) \circ (\Phi \otimes \Phi) \circ (L_u \otimes L_u) \circ \Delta \\ &= ((L_{\Phi(u)^{-1}} \circ \Phi \circ L_u) \otimes (L_{\Phi(u)^{-1}} \circ \Phi \circ L_u)) \circ \Delta\end{aligned}$$

and for $x \in C$:

$$\begin{aligned}\epsilon'(\hat{\Phi}(x)) &= \epsilon'(\Phi(u)^{-1}\Phi(ux)) = \epsilon'(\Phi(u))^{-1}\epsilon'(\Phi(ux)) \\ &= \epsilon(u)^{-1}\epsilon(ux) = \epsilon(u)^{-1}\epsilon(u)\epsilon(x) = \epsilon(x)\end{aligned}$$

whence $\hat{\Phi}$ is a morphism of topological graded coalgebras.

2. We compute

$$\begin{aligned}\Delta \circ \hat{D} &= \Delta \circ L_{u^{-1}} \circ D \circ L_u = L_{\Delta(u)^{-1}} \circ \Delta \circ D \circ L_u \\ &= (L_{u^{-1}} \otimes L_{u^{-1}}) \circ (D \otimes \mathbf{1} + \mathbf{1} \otimes D) \circ (L_u \otimes L_u) \circ D \\ &= (\hat{D} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{D}) \circ \Delta,\end{aligned}$$

and $\hat{D}$ is a graded coderivation of the same degree. Since $\hat{D}^2 = L_{u^{-1}} \circ D^2 \circ L_u$ it follows that $\hat{D}^2$ vanishes if D^2 vanishes.

3. We compute

$$\begin{aligned}\Delta \circ \tilde{D} &= \Delta \circ L_p \circ D - (-1)^{|p|\,|D|}\Delta \circ D \circ L_p \\ &= (L_p \otimes \mathbf{1} + \mathbf{1} \otimes L_p) \circ (D \otimes \mathbf{1} + \mathbf{1} \otimes D) \circ \Delta \\ &\qquad - (-1)^{|p|\,|D|}(D \otimes \mathbf{1} + \mathbf{1} \otimes D) \circ (L_p \otimes \mathbf{1} + \mathbf{1} \otimes L_p) \circ \Delta \\ &= ((D \circ L_p - (-1)^{|p|\,|D|}L_p) \otimes \mathbf{1} + \mathbf{1} \otimes (D \circ L_p - (-1)^{|p|\,|D|}L_p)) \circ \Delta \\ &= (\tilde{D} \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{D}) \circ \Delta. \end{aligned}$$

□

We apply this lemma to the situation $C = \mathcal{S}(\mathfrak{a}[2])[[h]]$, $C' = \mathcal{S}(\mathfrak{A}[2])[[h]]$, and Φ the formality map for a formal associative algebra $(\mathcal{A}, \mu_0)$. Let $\pi \in \mathfrak{a}[2]^0[[h]] = \mathfrak{a}^2$ a cohomology class such that $[\pi, \pi]_s = 0$, and let $\mu_* \in h\mathfrak{A}^2[[h]]$ such that $\Phi(e^{\bullet h\pi}) = e^{\bullet \mu_*}$, whence $\mu = \mu_0 + \mu_*$ is a formal associative deformation of μ_0 such that the class of μ_1 is given by π. Set

$$d' := e^{-hL_\pi} \circ d \circ e^{hL_\pi}, \tag{4.4}$$

$$(b + D)' := e^{-L_{\mu_*}} \circ (b + D) \circ e^{L_{\mu_*}}, \tag{4.5}$$

$$\Phi' := e^{-L_{\mu_*}} \circ \Phi \circ e^{hL_\pi} \tag{4.6}$$

the above Lemma 4.1 implies that d' is a graded coderivation of the topological coalgebra $S(\mathfrak{a}[2])[[h]]$ of square zero, that $(b + D)'$ is a graded coderivation of the topological coalgebra $S(\mathfrak{A}[2])[[h]]$ of square zero, and that Φ' is a morphism of graded topological coalgebras intertwining the codifferentials, i.e.

$$\Phi' \circ d' = (b + D)' \circ \Phi'. \tag{4.7}$$

Lemma 4.2 *There are the following expressions for the twisted coderivations*

1. $d' = d + h[\pi, \ ]_S$
2. $(b + D)' = b + [\mu_*, \ ]_G + D =: \mathbf{b} + D$

where $\mathbf{b}$ *is the Hochschild coboundary operator for the deformed multiplication* μ.

Proof 1. Writing $ad(\psi)\big(\chi\big)$ for $\psi \circ \chi - \chi \circ \psi$ where ψ is linear map of degree 0 we have

$$d' = e^{-h\ ad(L_\pi)}(d) = d + \sum_{r=1}^{\infty} \frac{(-h)^r}{r!} ad(L_\pi)^r(d).$$

By the above Lemma each of the terms in the exponential series is a graded coderivation of the cofree coalgebra $S(\mathfrak{a}[2])$ of degree 1. It is therefore sufficient to check the projections to $\mathfrak{a}[2]$ of the above identities: let $\xi_1, \dots, \xi_k \in \mathfrak{g}$:

$$\begin{aligned} &\mathsf{pr}_{\mathfrak{a}[2]}(\pi \bullet d(\xi_1 \bullet \cdots \bullet \xi_k) - d(\pi \bullet \xi_1 \bullet \cdots \bullet \xi_k)) \\ &= \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq 2, \\ 0 - [\pi, \xi_1]_s & \text{if } k = 1. \end{cases} \end{aligned}$$

We denote by $ad_s(\pi)$ the coderivation induced by $\xi \mapsto [\pi, \xi]_s$ which is also a derivation of the algebra $\mathcal{S}\mathfrak{a}[2]$ at the same time. Furthermore

$$ad(L_\pi)^2(d) = -L_\pi \circ ad_s(\pi) + ad_s(\pi) \circ L_\pi = L_{[\pi,\pi]_s} = 0$$

whence the higher order terms of the exponential series vanish.

2. Since b is also a graded derivation of the graded commutative algebra $S(\mathfrak{A}[2])$, we get $-L_{\mu_*} \circ b + b \circ L_{\mu_*} = L_{b\mu_*}$, and since $\mathcal{S}(\mathfrak{A}[2])$ is graded commutative the higher order terms of the series vanish, and one gets

$$b' = e^{-ad(L_{\mu*})}(b) = b + L_{b\mu_*}.$$

Moreover, by an analogous reasoning as above we get

$$D' = e^{-ad(L_{\mu*})}(D) = D + ad_G(\mu_*) + \frac{1}{2} L_{[\mu_*,\mu_*]_G}$$

where we denoted by $ad_G(\mu_*)$ the coderivation of $\mathcal{S}(\mathfrak{A}[2])$ induced by $\xi \mapsto [\mu_*, \xi]_G$ which also is a derivation of the algebra $\mathcal{S}(\mathfrak{A}[2])$. Hence

$$(b + D)' = b' + D' = b + ad_G(\mu_*) + D + L_{b\mu_*} + \frac{1}{2} L_{[\mu_*,\mu_*]_G} = ad_G(\mu) + D + 0 = \mathbf{b} + D$$

thanks to the Maurer–Cartan equation for μ_*. □

5 Deformation Quantization of Polynomial Algebras

In this section we recall the important particular case where $(\mathcal{A}, \mu_0)$ is the symmetric algebra $\mathcal{S}E$ where E is a finite dimensional real or complex vector space which is ungraded ($\mathbb{Z}$-graded of degree 0). A particular case of the Hochschild–Kostant–Rosenberg Theorem, which is in fact is the particular case of an Abelian Lie algebra in the Cartan–Eilenberg Theorem 2.1, shows that for each nonnegative integer k there is

$$\mathfrak{a}^k = \mathbf{H}_H^k(\mathcal{A}, \mathcal{A}) \cong \mathcal{S}E \otimes \Lambda^k E^* =: \mathcal{T}_{\text{poly}}^k$$

where the latter space is also called the space of algebraic *polyvector fields* of rank k. We set $\mathcal{T}_{\text{poly}} := \oplus_{k=0}^{\infty} \mathcal{T}_{\text{poly}}^k$. For the computations, we shall use the canonical identification of $\mathcal{S}E$ with the algebra of all polynomial functions on the dual space $V := E^*$. Let $e_1, \ldots, e_n$ be a basis of V, and let $e^1, \ldots, e^n$ be the dual basis. we can regard any $f \in \mathcal{S}E$ as a polynomial in the coordinates $x^1, \ldots, x^n$ (upon writing each $x \in V$ as $x = \sum_{i=1}^n x^i e_i$). The polyvector fields are now polynomial functions with values in ΛV. Using partial derivatives $\partial/\partial x^i$ in $\mathcal{S}E$ and interior products ι_{e^i} with respect to the dual basis in ΛV, we can write the projected Gerstenhaber bracket $[\ ,\ ]_s$ in its classical form an algebraic *Schouten bracket*

$$[\xi, \eta]_s := \sum_{i=1}^n \iota_{e^i} \xi \wedge \frac{\partial \eta}{\partial x^i} - (-1)^{(|\xi|-1)(|\eta|-1)} \sum_{i=1}^n \iota_{e^i} \eta \wedge \frac{\partial \xi}{\partial x^i}. \tag{5.1}$$

As the Gerstenhaber bracket, the Schouten bracket defines a graded Lie bracket on $\mathcal{T}_{\text{poly}}[1]$, i.e. for homogeneous elements $\xi, \eta, \zeta \in \mathcal{T}_{\text{poly}}$ one has

$$[\xi, \eta]_s = -(-1)^{(|\xi|-1)(|\eta|-1)} [\eta, \xi]_s, \tag{5.2}$$

$$\begin{aligned} 0 &= (-1)^{(|\xi|-1)(|\zeta|-1)} [\xi, [\eta, \zeta]_s]_s + (-1)^{(|\eta|-1)(|\xi|-1)} [\eta, [\zeta, \xi]_s]_s \\ &\quad + (-1)^{(|\zeta|-1)(|\eta|-1)} [\zeta, [\xi, \eta]_s]_s. \end{aligned} \tag{5.3}$$

On the other hand there is the pointwise exterior multiplication $\wedge$ which makes $\mathcal{T}_{\rm poly} = \mathcal{S}(V \oplus V[-1])$ a graded commutative algebra. The Schouten bracket and the exterior multiplication are compatible by the graded Leibniz rule

$$[\xi, \eta \wedge \zeta]_s = [\xi, \eta]_s \wedge \zeta + (-1)^{(|\xi|-1)|\eta|} \eta \wedge [\xi, \zeta]_s. \tag{5.4}$$

Note the following particular cases for the Schouten bracket where $\mathcal{T}^1_{\rm poly}$ is the subspace of all vector fields (i.e. derivations of $\mathcal{S}E = \mathcal{T}^0_{\rm poly}$) where [,] denotes the usual Lie bracket (i.e. commutator) of derivations:

$$\begin{aligned}
&[f, g]_s = 0 && \text{for all } f, g \in \mathcal{S}V,\\
&[X, f]_s = X(f) = \sum_{i=1}^n X^i \frac{\partial f}{\partial x^i} && \text{for all } X \in \mathcal{T}^1_{\rm poly},\ f \in \mathcal{S}V,\\
&[X, Y]_s = [X, Y] = \sum_{i=1}^n \left(X^i \frac{\partial Y}{\partial x^i} - Y^i \frac{\partial X}{\partial x^i} \right) && \text{for all } X, Y \in \mathcal{T}^1_{\rm poly}.
\end{aligned}$$

5.1 (Linear) Poisson Structures

Let $\pi = (1/2) \sum_{i,j} \pi^{ij} e_i \wedge e_j \in \mathcal{T}^2_{poly}$ a so-called bivector field. If in addition π satisfies

$$[\pi, \pi]_s = 0,$$

or in coordinates

$$\sum_{h=1}^n \left(\pi^{ih} \frac{\partial \pi^{jk}}{\partial x^h} + \pi^{jh} \frac{\partial \pi^{ki}}{\partial x^h} + \pi^{kh} \frac{\partial \pi^{ij}}{\partial x^h} \right) = 0$$

it is called a *Poisson structure*. Then the bilinear map $\{\ ,\ \}_\pi : \mathcal{S}V \times \mathcal{S}V \to \mathcal{S}V$ defined by

$$\{f, g\}_\pi =: -[[\pi, f]_s, g]_s = \sum_{i,j=1}^n \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

equips the associative commutative algebra $\mathcal{A} := \mathcal{S}V$ with an *Poisson bracket*, i.e. there are the Jacobi and the Leibniz identities for all $f, g, h \in \mathcal{A}$

$$\begin{aligned}
&\{f, \{g, h\}_\pi\}_\pi + \{g, \{h, f\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi = 0,\\
&\{h, fg\}_\pi = \{h, f\}_\pi g + f\{h, g\}_\pi.
\end{aligned}$$

In general, a commutative associative algebra satisfying the preceding identities is called a *Poisson algebra*. Poisson structures are very important in deformation theory and many other contexts as integrable systems or representation theory [33].

Note that the operator $\delta_\pi := [\pi, -]_s$ defines a complex on the space of all polyvector fields whose cohomology is the so-called *Poisson cohomology*.

Let $(E = \mathfrak{g}, [\ ,\ ])$ be a finite dimensional Lie algebra over $\mathbb{K}$ and $V = \mathfrak{g}^*$ its algebraic dual. The Lie algebra structure [,] of $\mathfrak{g}$ can canonically be considered as an element $\pi_0 \in \mathcal{S}^1\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^* = \mathcal{T}^2_{\rm poly}$ and is a Poisson structure thanks to the Jacobi identity for the

Lie bracket [,]. In terms of the structure constants $C_i^{jk} := \langle [e^j, e^k], e_i\rangle$ of $\mathfrak{g}$ the bivector field π_0 takes the form

$$\pi_0(x) = \frac{1}{2}\sum_{i,j,k} C_i^{jk} x^i e_j \wedge e_k. \tag{5.5}$$

It is easily computed that the complex $(\mathcal{T}_{\mathrm{poly}}, \delta_{\pi_0})$ is identical to the Chevalley–Eilenberg complex of $(\mathbf{C}_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}), \delta_{CE})$, hence the Poisson cohomology is given by the Lie algebra cohomology.

5.2 Formality Maps for Polynomial Algebras and Deformation Quantization

From now we assume that the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. In his celebrated work [29] Kontsevich gave an explicit formula for a formality map $\Phi : \mathcal{S}(\mathfrak{a}[2]) \to \mathcal{S}(\mathfrak{A}[2])$. Since it is a morphism of graded symmetric coalgebras it is enough to specify its $\mathfrak{A}[2]$-component, that is the family of degree zero linear maps $\Phi_n : \mathcal{S}^n(\mathfrak{a}[2]) \to \mathfrak{A}[2]$ for each positive integer n (which is called $\mathcal{U}_n$ in [29]). Take n polyvector fields $\xi_1, \dots, \xi_n \in \mathfrak{a}$. Let N_{der} be the sum of all the ranks of the ξ_i, i.e. $N_{\mathrm{der}} := |\xi_1| + \cdots + |\xi_n|$. Since $\xi_1 \bullet \cdots \bullet \xi_n$ is of degree $N_{\mathrm{der}} - 2n$ in $\mathcal{S}^n(\mathfrak{a}[2])$ it follows that $\Phi_n(\xi_1 \bullet \cdots \bullet \xi_n)$ lies in $\mathfrak{A}[2]^{N_{\mathrm{der}}-2n} = \mathfrak{A}^{N_{\mathrm{der}}-2n+2}$ which is the space of all Hochschild cochains having $m := N_{\mathrm{der}} - 2n + 2$ arguments in $\mathcal{S}E$. Kontsevich encodes the information in planar graphs having n vertices of the first kind, m vertices of the second kind and $N_{\mathrm{der}} = m + 2n - 2$ edges, see page 25 of [29]. The Hochschild n-cochain $\Phi_n(\xi_1 \bullet \cdots \bullet \xi_n)$ is a finite real linear combinations over expressions of the following kind:
First, write each polyvector field ξ of rank $k = |\xi|$ in a basis as

$$\xi = (1/k!) \sum_{i_1,\dots,i_k=1}^{\dim E} \xi^{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

with components $\xi^{i_1\cdots i_k} \in \mathcal{S}E$. Then define the n integers $k_1, \dots, k_n$ by

$$k_i = |\xi_1| + \cdots + |\xi_i|,$$

choose $n + m - 1$ integers $r_1, \dots, r_{n+m-1}$ such that

$$0 \le r_1 \le r_2 \le \cdots \le r_{n+m-1} \le N_{\mathrm{der}},$$

and a permutation σ of $\{1, \dots, N_{\mathrm{der}}\}$. Calling this choice Γ, Kontsevich defines for all $f_1, \dots, f_m \in \mathcal{S}E$

$$\begin{aligned}
&\Phi_\Gamma(\xi_1 \bullet \cdots \bullet \xi_n)(f_1, \dots, f_m) \\
&\quad = \sum_{i_1,\dots,i_{N_{\mathrm{der}}}=1}^{\dim E} \frac{\partial^{r_1} \xi_1^{i_1\cdots i_{k_1}}}{\partial x^{i_{\sigma(1)}} \cdots x^{i_{\sigma(r_1)}}} \cdots \frac{\partial^{r_n - r_{n-1}} \xi_n^{i_{k_{n-1}+1}\cdots i_{k_n}}}{\partial x^{i_{\sigma(r_{n-1}+1)}} \cdots x^{i_{\sigma(r_n)}}} \\
&\qquad \times \frac{\partial^{r_{n+1}-r_n} f_1}{\partial x^{i_{\sigma(r_n+1)}} \cdots x^{i_{\sigma(r_{n+1})}}} \cdots \frac{\partial^{N_{\mathrm{der}} - r_{n+m-1}} f_m}{\partial x^{i_{\sigma(r_{n+m-1}+1)}} \cdots x^{i_{\sigma(N_{\mathrm{der}})}}},
\end{aligned} \tag{5.6}$$

and Φ_n is a real linear combination over all possible choices Γ which are encoded by the Kontsevich graphs:

$$\Phi_n = \sum_\Gamma w_\Gamma \Phi_\Gamma.$$

where the real numbers w_Γ are the Kontsevich weights given by integrals over configuration spaces. Roughly speaking: one distributes N_{der} partial derivatives over the $n+m$ symbols consisting of the n polyvector fields and the m polynomials and summing the N_{der} upper and lower indices in some chosen fashion.

In particular, for $n=1$ we get $m=|\xi|=N_{\mathrm{der}}$, and Φ_1 is shown to be the usual HKR-map.

Moreover, for a Poisson structure $\pi\in\mathfrak{a}^2$ the formal associative deformation μ of the commutative multiplication μ_0 of $\mathcal{S}E$ by μ_* is given by

$$\mu=\mu_0+\sum_{r=1}^{\infty}\frac{h^r}{r!}\Phi_r(\pi^{\bullet r}). \tag{5.7}$$

Note that for each r the number of derivatives N_{der} is equal to $2r$ and m, the number of functions is equal to 2. In the theory of *deformation quantization*, the formal deformation μ defined by π is called a *star-product*, see [1, 3, 5, 10, 25]. In that theory, which gives a sort of asymptotic formulation of a quantum mechanical observable algebra, the commutative associative algebra is taken to be the ring of all smooth $\mathbb{K}$-valued functions on a smooth (Poisson) manifold. Kontsevich's techniques allow to prove that these deformations exist for any Poisson structure.

6 Deformation and Formality of Universal Enveloping Algebras

In this section we shall consider a linear Poisson structure π_0 defined by the bracket of a finite-dimensional Lie algebra $\mathfrak{g}=E$.

6.1 Convergence of the Twisted Kontsevich Formality Map on UEAs

Theorem 6.1 *Let π_0 be a linear Poisson structure with respect to the finite-dimensional Lie algebra $(\mathfrak{g},[\ ,\ ])$, and let Φ be Kontsevich's formality map.*

1. *Then the twisted formality map $\Phi'=e^{-L_{\mu_*}}\circ\Phi\circ e^{hL_{\pi_0}}$ of Lemma 4.1 takes the following form for $n\geq 1$, $\xi_1,\ldots,\xi_n\in\mathfrak{a}$ and $f_1,\ldots,f_m\in\mathcal{S}\mathfrak{g}$ where $m=|\xi_1|+\cdots+|\xi_n|-2n+2$:*

$$\Phi'_n(\xi_1\bullet\cdots\bullet\xi_n)(f_1,\ldots,f_m)=\sum_{r=0}^{\infty}\frac{h^r}{r!}\Phi_{n+r}(\pi_0^{\bullet r}\bullet\xi_1\bullet\cdots\bullet\xi_n)(f_1,\ldots,f_m). \tag{6.1}$$

2. *For each choice of $\xi_1,\ldots,\xi_n$ and $f_1,\ldots,f_m$ the above equation (6.1) is only a finite sum, hence converges for $h=1$.*
3. *Denoting by the same symbol Φ' the twisted formality map at $h=1$ we have that Φ' defines an L_∞-morphism, i.e. a morphism of graded commutative differential coalgebras of the coalgebra over the Chevalley–Eilenberg complex $(\mathcal{S}(\mathfrak{a}[2]),\delta_{CE}+d)$ of the Lie algebra $\mathfrak{g}$ to the coalgebra over the Hochschild complex $(\mathcal{S}(\mathfrak{A}[2]),\mathbf{b}+D)$ of the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$.*

Proof 1. Upon projecting on $\mathfrak{A}[2]$ we get rid of the exponential of μ_* for $n\geq 1$ and immediately get the announced formula.

2. Let $d_1,\ldots,d_{n+m}$ be the polynomial degrees of $\xi_1,\ldots,\xi_n,f_1,\ldots,f_m$. Since π is of polynomial degree one it follows that for each nonnegative integer r in formula (6.1) there are $2r+N_{\mathrm{der}}$ partial derivatives distributed over a total polynomial degree of $r+d_1+\cdots+$

d_{n+m}. Clearly, for each r which is greater or equal to $d_1 + \cdots + d_{n+m} - N_{\mathrm{der}} + 1$ the term in the sum vanishes which shows the convergence.

3. In the paper [2] we have shown that the deformed multiplication $\mu = \mu_0 + \mu_*$ converges for $h = 1$ (which also follows from a similar power counting argument for equation (5.7) as in 2.), and that this convergent multiplication is isomorphic to the multiplication in the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. The rest of the statement is just a reformulation of the formulas in Lemma 4.2. □

6.2 Formality in Stages

The preceding Theorem 6.1 allows us to prove formality for Universal enveloping algebras once the Chevalley–Eilenberg cohomology admits an L_∞-morphism into the Chevalley–Eilenberg complex:

Theorem 6.2 *Let* $(\mathfrak{g}, [\ ,\ ])$ *be a finite-dimensional Lie-algebra. Assume that there is an* L_∞*-morphism* Φ'' *of the Chevalley–Eilenberg cohomology* $\mathfrak{a}' := \mathrm{H}_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$ *equipped with the induced Schouten bracket* $[\ ,\ ]'_s$ *into the Chevalley–Eilenberg complex* $\mathfrak{a} := \mathbf{C}_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$ *such that* $\Phi''_1 : \mathfrak{a}' \to \mathfrak{a}$ *is injective into the space of cocycles* (*a HKR map in this situation*). *Then the universal enveloping algebra* $\mathcal{U}\mathfrak{g}$ *of* $\mathfrak{g}$ *is formal.*

Proof By assumption, Φ'' is a morphism of graded cocommutative differential coalgebras $(\mathcal{S}(\mathfrak{a}'[2]), d')$ into $(\mathcal{S}(\mathfrak{a}[2]), d + \delta_{CE})$ where d' is the graded coderivation of degree 1 induced by the induced Schouten bracket. It is clear that the composition $\Phi' \circ \Phi'' : (\mathcal{S}(\mathfrak{a}'[2]), d') \to (\mathcal{S}(\mathfrak{A}'[2]), \mathbf{b} + D)$ is an L_∞-morphism. Since the latter is built over the Hochschild complex of $\mathcal{U}\mathfrak{g}$ and the Chevalley–Eilenberg cohomology is isomorphic to the Hochschild cohomology by the Cartan–Eilenberg Theorem 2.1. The Theorem follows. □

6.3 Examples

6.3.1 Abelian Lie Algebras

In case the Lie algebra $\mathfrak{g}$ is Abelian, then there is nothing to prove since the Chevalley–Eilenberg differential is zero, and formality of $\mathcal{U}\mathfrak{g} = \mathcal{S}\mathfrak{g}$ is the content of the Kontsevich formality theorem.

6.3.2 Lie Algebra of the Affine Group of $\mathbb{K}^m$

Theorem 6.3 *Let* $\mathfrak{g}$ *be the affine Lie algebra i.e. the semidirect sum*

$$\mathfrak{gl}(m, \mathbb{K}) \oplus \mathbb{K}^m.$$

Then $\mathcal{U}\mathfrak{g}$ *is formal.*

Proof Thanks to Proposition 3.2 the Chevalley–Eilenberg cohomology of the affine Lie algebra is given by the scalar cohomology of the matrix Lie algebra $\mathfrak{gl}(m, \mathbb{K})$. This means that the classes can be represented by cocycles in the Chevalley–Eilenberg complex which are in $\Lambda\mathfrak{g}^*$, that is by constant polyvector fields annihilated by δ_{CE}. Since all constant polyvector fields have zero Schouten brackets with themselves, it follows that the induced Schouten bracket on the cohomology space is zero, and that any HKR-map Φ''_1 of the cohomology in the subalgebra of constant polyvector fields is a morphism of graded Lie algebras. Therefore the coalgebra map Φ'' induced by Φ''_1 is the desired L_∞-morphism of the assumption of Theorem 6.2. This proves the formality of $\mathcal{U}\mathfrak{g}$. □

6.4 Construction of Nonrigid Universal Enveloping Algebras

In order to generalize the two criteria for (3.1) and (3.2) for nonrigidity of Universal Enveloping algebras, we can use the preceding formality results:

Since an L_∞-morphism maps so-called Maurer–Cartan elements, i.e. elements π_* of $t\mathfrak{a}^2[[t]]$ such that $(d+\delta_{CE})(e^{\bullet\pi_*})=0$ or

$$\delta_{CE}\pi_* + \frac{1}{2}[\pi_*,\pi_*]_s = 0, \tag{6.2}$$

to Maurer–Cartan elements $\nu_* \in \mathfrak{A}^2[[t]]$, i.e. $\Phi(e^{\bullet\pi_*}) = e^{\bullet\nu_*}$, it follows that a formal deformation $\pi = \pi_0 + \pi_*$ of the linear Poisson structure π_0 gives rise to a formal deformation $\nu_0 + \nu_*$ of the multiplication ν_0 of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. This implies the major part of the following nonrigidity theorem proved in [2]:

Theorem 6.4 ([2]) *Let $\pi \in \mathcal{S}\mathfrak{g} \otimes \Lambda^2\mathfrak{g}^*[[t]]$ be a formal polynomial Poisson deformation of the linear Poisson structure on $\mathfrak{g}^*$ such that the first order term π_1 is a nontrivial Chevalley–Eilenberg 2-cocycle of $\mathfrak{g}$. Then there exists a nontrivial formal associative deformation of $\mathcal{U}\mathfrak{g}$, hence $\mathfrak{g}$ is not strongly rigid.*

7 Examples of Deformations of Linear Poisson Structures

In this section, we construct quadratic deformations of linear Poisson structure associated to some solvable Lie algebras and discuss the strong rigidity of the corresponding Lie algebras.

Let $\mathbb{K}$ be the complex field. Let $\mathfrak{g}$ be a finite dimensional decomposable solvable Lie algebra, i.e. $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical and $\mathfrak{t}$ is an exterior torus of derivations in Malcev's sense; that is $\mathfrak{t}$ is an Abelian subalgebra of $\mathfrak{g}$ such that adx is semisimple for all $x \in \mathfrak{t}$. This class of solvable Lie algebra contains the non semisimple rigid Lie algebras [6]. The classification of n-dimensional rigid Lie algebras is known up to $n \le 8$ [24].

Dimension 2. There is one isomorphism class, namely the Lie algebra $\mathfrak{r}_2$ which is strongly rigid (see Proposition 3.1). It follows that the linear Poisson structure associated to Lie algebra $\mathfrak{r}_2$ is rigid.

Dimension 3. There are no solvable rigid Lie algebras.

Dimension 4. There is only one rigid Lie algebras, $\mathfrak{r}_2 + \mathfrak{r}_2$. Since the torus is 2-dimensional, then according to corollary (3.1) this algebra is not strongly rigid.

Dimension 5. There is only one rigid class with 2-dimensional torus. There is no strongly rigid Lie algebra.

In the following we construct nontrivial deformations of the linear Poisson structure associated to the solvable Lie algebra $\mathfrak{g}_5^1$ defined, with respect to a basis $\{x, x_0, y_1, y_2, y_3\}$, by the following nontrivial skewsymmetric brackets:

$$[x, y_i] = i y_i, \quad i = 1, 2, 3,$$
$$[x_0, y_i] = y_i, \quad i = 2, 3,$$
$$[x_1, y_2] = y_3.$$

The 2-dimensional torus being generated by x, x_0. We consider the vectors x_i as a variables and we denote by $\frac{\partial}{\partial x_i}$ the elements of the dual basis.

Proposition 7.1 *Let $P \in \mathcal{S}\mathfrak{g}_5^1 \otimes \wedge^2(\mathfrak{g}_5^1)^*$, $P = P_0 + tP_1$ where P_0 is a linear Poisson structure associated to the Lie algebra $\mathfrak{g}_5^1$ and $P_1 \in \mathcal{S}\mathfrak{g}_5^1 \otimes \wedge^2(\mathfrak{g}_5^1)^*$ is defined by $(\alpha, \beta, \gamma \in \mathbb{C}^*)$:*

$$P_1 = \frac{\alpha\beta}{\gamma} x_0 x_3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x_2} - \beta x_0 x_1 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x_0} + \alpha x_0 x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \gamma x_0 x_1 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} - (\gamma x_0 x_2 + \beta x_0 x_3) \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_2}. \quad (7.1)$$

Then $[P, P]_s = 0 = [P_0, P_1]_s$ and the cohomology class of P_1 is not 0.
Therefore, P is a nontrivial deformation of P_0 and $\mathfrak{g}_5^1$ is not strongly rigid.

Proof By a direct calculation, we show $[P, P]_s = 0 = [P_0, P]_s$.
Let $\mathcal{S}_2\mathfrak{g}_5^1$ be the space of quadratic polynomials with variables x, x_0, x_1, x_2, x_3. The 2-cochains quadratic space is $\mathcal{S}^2\mathfrak{g}_5^1 \otimes \wedge^2(\mathfrak{g}_5^1)^*$ and the quadratic 1-cochains space is $\mathcal{S}^2\mathfrak{g}_5^1 \otimes (\mathfrak{g}_5^1)^*$. We show that there is no element A of $\mathcal{S}^2\mathfrak{g}_5^1 \otimes (\mathfrak{g}_5^1)^*$ such that $\delta_{CE} A = P_1$. By Theorem 6.4 we deduce that the Lie algebra is not strongly rigid. □

Dimension 6. There are 3 isomorphism classes of 6-dimensional rigid solvable Lie algebras. Only one has a one-dimensional torus. Let us consider this Lie algebra, it is denoted in [24] by $\mathfrak{t}_1 \oplus \mathfrak{n}_{5,6}$. Setting the basis $\{x_0, x_1, x_2, x_3, x_4, x_5\}$ the Lie algebra is defined by the following nontrivial skewsymmetric brackets

$$[x_0, x_i] = i x_i, \quad i = 1, \dots, 5, \quad (7.2)$$

$$[x_1, x_i] = x_{i+1}, \quad i = 2, 3, 4, \quad (7.3)$$

$$[x_2, x_3] = x_5. \quad (7.4)$$

In the following we construct nontrivial deformations of the linear Poisson structure associated to the Lie algebra $\mathfrak{g} = \mathfrak{t}_1 \oplus \mathfrak{n}_{5,6}$.

Proposition 7.2 *Let $P \in \mathcal{S}\mathfrak{g} \otimes \wedge^2\mathfrak{g}^*$, $P = P_0 + tP_1$ where P_0 is a linear Poisson structure associated to the Lie algebra $\mathfrak{g}$ and $P_1 \in \mathcal{S}\mathfrak{g} \otimes \wedge^2\mathfrak{g}^*$ defined by $(\alpha, \beta, \gamma \in \mathbb{C}^3 \setminus \{(0,0,0)\})$:*

$$P_1 = \beta x_2^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + \gamma \left(-x_2 x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + x_2 x_5 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \right) + \alpha x_1 x_5 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4}.$$

Then $[P, P]_s = 0 = [P_0, P_1]_s$ and the cohomology class of P_1 is not 0.
Thus, P is a nontrivial deformation of P_0 and $\mathfrak{g}$ is not strongly rigid.

Proof Straightforward computation and Theorem 6.4. □

Dimension 7. There are 8 isomorphism classes of 7-dimensional rigid solvable Lie algebras. Only 3 have a one-dimensional torus. Let us consider one of these Lie algebra which is denoted by $\mathfrak{g}_7^1$ in [24].

Setting the basis $\{x_0, x_1, \dots, x_6\}$ the Lie algebra $\mathfrak{g}_7^1$ is defined by the following nontrivial skewsymmetric brackets

$$[x_0, x_i] = i x_i, \quad i = 1, \dots, 6,$$

$$[x_1, x_i] = x_{i+1}, \quad i = 2, \dots, 5,$$

$$[x_2, x_i] = x_{i+2}, \quad i = 2, 4.$$

In the following we give a nontrivial deformation of the linear Poisson structure associated to the Lie algebra $\mathfrak{g}_7^1$.

Proposition 7.3 *Let* $P \in \mathcal{S}\mathfrak{g}_7^1 \otimes \wedge^2(\mathfrak{g}_7^1)^*$, $P = P_0 + tP_1$ *where* P_0 *is a linear Poisson structure associated to the Lie algebra* $\mathfrak{g}7^1$ *and* $P_1 \in \mathcal{S}\mathfrak{g}_7^1 \otimes \wedge^2(\mathfrak{g}_7^1)^*$ *defined by* $(\alpha \neq 0)$:

$$P_1 = 2\alpha x_4 x_5 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} + \alpha x_5^2 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_2}.$$

Then $[P, P]_s = 0 = [P_0, P_1]_s$ *and the cohomology class of* P_1 *is not* 0.

Therefore, P *is a non trivial deformation of* P_0.

Thus, P *is a nontrivial deformation of* P_0 *and* $\mathfrak{g}_7^1$ *is not strongly rigid*.

Proof Straightforward computation and Theorem 6.4. □

7.1 Classification of Strongly Rigid Lie Algebras of Small Dimension

One may deduce from the previous section the following classification of strongly n-dimensional solvable strongly rigid Lie algebra for $n \leq 6$.

Proposition 7.4 *There is only one* n*-dimensional solvable strongly rigid Lie algebra for* $n \leq 6$, *namely the* 2*-dimensional Lie algebra* $\mathfrak{r}_2$.

Then we have the following list, obtained in [2], of strongly rigid complex Lie algebras up to dimension 6 as follows: it suffices to look at rigid Lie algebras (see e.g. [24] for some list), and we ruled out two 6-dimensional Lie algebras by constructing quadratic nontrivial deformations of their linear Poisson structures and applying Theorem 6.4. We get:

$$\{0\};\quad \mathbb{C};\quad \mathfrak{aff}(1,\mathbb{C});\quad \mathfrak{sl}(2,\mathbb{C});\quad \mathfrak{gl}(2,\mathbb{C});\quad \mathfrak{aff}(1,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C});$$
$$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C});\quad \mathfrak{aff}(2,\mathbb{C}).$$

One may have also the following consequence: given a Poisson structure, if there exists a formal isomorphism such that this Poisson structure is isomorphic to its linear part then one says that this Poisson structure is linearizable. This problem was formulated first by A. Weinstein (based on considerations by Sophus Lie) [13, 34]. Using the Theorem 6.4, one may deduce:

Proposition 7.5 *Every Poisson structure which is a deformation of linear Poisson structure of* n*-dimensional strong rigid solvable Lie algebra is linearizable*.

It follows that every Poisson structure which is a deformation of linear Poisson structure of n-dimensional solvable Lie algebra, with $3 \leq n \leq 6$, is linearizable. The Poisson structure $P_0 + P_1$ (defined in Proposition 7.2) is not linearizable.

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